

On the number of colored Birch and Tverberg partitions

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Abstract

Recently, there have been new developments concerning the colored Tverberg problem, see Blagojević, Matschke, and Ziegler [2], [3], and Matoušek, Tancer, and Wagner [6]. In the uncolored version of Tverberg's theorem, lower bounds for the number of Birch partitions led to new lower bounds for the number of Tverberg partitions, see Hell [5] for details. Therefore, we introduce the concept of colored Birch partitions, and come up with lower bounds. These bounds carry over to the first non-trivial lower bounds for the number of colored Tverberg partitions.

1 Introduction

The first colored Tverberg theorem is due to Bárány and Larman [1], see Ziegler [8] for the complete story. Here we start from the following generalization due to Blagojević, Matschke, and Ziegler [3, Theorem 1.3, Case $k=0$].

Theorem 1. *Let $d \geq 1$, $r \geq 2$ prime, $N := (d+1)(r-1)$, and $f : \Delta_N \rightarrow \mathbb{R}^d$ continuous, where the $N+1$ vertices of Δ_N have $d+2$ different colors, and the color classes satisfy: $|C_0| = |C_1| = \dots = |C_d| = r-1$ and $|C_{d+1}| = 1$. Then the simplex Δ_N has r disjoint rainbow faces F_1, F_2, \dots, F_r whose images under f have a non-empty intersection:*

$$\bigcap_{i=1}^r f(F_i) \neq \emptyset.$$

Here *rainbow* means that every color occurs at most once. In the following, we focus on its affine version. In this case, one can think of $N+1$ colored points in \mathbb{R}^d satisfying the above color condition which can be partitioned into r rainbow partition blocks F_1, F_2, \dots, F_r such that their convex hulls intersect:

$$\bigcap_{i=1}^r \text{conv}(F_i) \neq \emptyset.$$

Theorem 1 settles the existence of one (!) partition, but unlike the uncolored case not much is known for *How many of these colored partitions exist?* Matschke and Ziegler ask this question for the first time in [7], as part of their Smarties[®]-problem.

Assuming that the points are in general position, the partition blocks thus consist of at most $d+1$ points. A solution could be a single point that lies in the convex hulls of $r-1$ many $(d+1)$ -element sets. The other extreme case would be d partition blocks of exactly (!) d points each, intersecting in a single point, plus $r-d$ many $(d+1)$ -element sets, that all contain the intersection point, here $d \leq r$. In all cases, we have at least $r-d-1$ many $(d+1)$ -element sets $F_1, F_2, \dots, F_{r-d-1}$ which (i) contain a fixed point in their convex hull, and which (ii) are rainbow sets in the

following way: each contains each of the colors $0, 1, \dots, d$ exactly once, or short: $|F_i \cap C_j| = 1$ for all $0 \leq j \leq d$, and all $1 \leq i \leq r - d - 1$.

This observation leads to the concept of colored Birch partitions. For this, let $p \in \mathbb{R}^d$ be a point, and $k \geq 1$ a natural number. Given a set X of $k(d+1)$ colored points in \mathbb{R}^d of $d+1$ different colors such that each color class C_0, C_1, \dots, C_d contains exactly k points, we call a partition F_1, F_2, \dots, F_k a *colored Birch partition of X to the point p* , if each block F_i contains exactly $d+1$ points, uses every color exactly once, and contains p in its convex hull. Let $\text{cBP}_k(X)$ be the number of all unordered colored Birch partitions of X to p . Here *unordered* means that two partitions are regarded as the same if one can be obtained from the other by a permutation of the k partition blocks. The partitions in the previous paragraph are examples of colored Birch partitions to the single point resp. the intersection point. Placing p outside the convex hull of X one gets $\text{cBP}_k(X) = 0$. See also Figure 1 for an example for $k = 4$ with $\text{cBP}_k(X) = 2$. By definition: $\text{cBP}_k(X) \leq \text{BP}_k(X)$, where $\text{BP}_k(X)$ is the number of (uncolored !) Birch partitions, see Hell [5] for more properties.

Let us formulate our main results. For this, a set of points is in *general position* if no $k+2$ points are on a common k -dimensional affine subspace.

Theorem 2. *For $d \geq 1$, let $p \in \mathbb{R}^d$ be a point, and k a natural number. For a given set X of $k(d+1)$ colored points in \mathbb{R}^d of $d+1$ different colors such that each color class C_0, C_1, \dots, C_d contains exactly k points, and $X \cup \{p\}$ in general position, the number of colored Birch partitions $\text{cBP}_k(X)$ has the following four properties:*

- (i) $\text{cBP}_k(X)$ is even for $k \geq d+2$.
- (ii) $\text{cBP}_k(X) > 0 \implies \text{cBP}_k(X) \geq \lceil \frac{k}{2} \rceil! \cdot \lfloor \frac{k}{2} \rfloor!$ for $d = 1$.
- (iii) $\text{cBP}_k(X) > 0 \implies \text{cBP}_k(X) \geq 8 \cdot 3^{k-6}$ for $d = 2$ and $k \geq 7$.
- (iv) $\text{cBP}_k(X) > 0 \implies \text{cBP}_k(X) \geq 2^{k-d-1}$ for $d \geq 2$ and $k \geq d+2$.

Property (ii) is an easy exercise. In the planar setting, a point configuration can be represented as a colored word of length $3k$ on the alphabet $\{+, -\}$: Choose a line through the origin. This line hits at most one point from X , and it divides the plane into two half-spaces. Choose one of the two half-spaces. Then sweep a line through the origin over the chosen half-space counter-clockwise. The ray hits all points exactly once, and the sweeping leads to a linear order on the points in X . This determines a colored word of length $3k$ on the alphabet $\{+, -\}$ in the following way: Write for every point of X the letter $+$ when the line hits a point in the chosen half-space, and the $-$ in the other case. While writing the letters, keep for each letter track of its color.

Every possibility of partitioning colored word of length $3k$ into k colored sub-words of the form $+-+$, or $-+-$ corresponds one-to-one to a colored Birch partition of X . One can check that the alternating word $+-+-+-$ of length 9 with a cyclic coloring $0, 1, 2, 0, 1, 2, 0, 1, 2$ corresponds to a colored point configuration with $\text{cBP}_3(X) = 3$ showing that $k > d+1$ is necessary. Namely, one partition is $\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}$, where the letters are numbered from left to right. The other two are $\{0, 1, 8\}, \{2, 3, 4\}, \{5, 6, 7\}$, and $\{0, 7, 8\}, \{1, 2, 3\}, \{4, 5, 6\}$.

Computer experiments for small k in dimensions 2, 3, and 4 suggest that the lower bounds are tight.

Theorem 2 implies analogous properties for the number of colored Tverberg partitions.

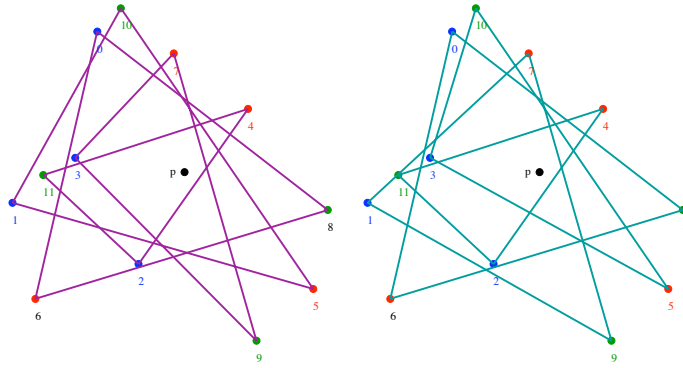


Figure 1: A planar example for $k = 4$ with $\text{cBP}_k(X) = 2$.

Theorem 3. *Let $d \geq 1$, $r \geq 2$ prime, $N := (d+1)(r-1)$, and $f : \Delta_N \rightarrow \mathbb{R}^d$ affine and in general position, where the $N+1$ vertices of Δ_N have $d+2$ different colors, and the color classes satisfy: $|C_0| = |C_1| = \dots = |C_d| = r-1$ and $|C_{d+1}| = 1$. Then the number of (unordered) colored Tverberg partitions $T(f)$ satisfies the following four properties:*

- (i) $T(f)$ is even for $r \geq 2d+3$.
- (ii) $T(f) \geq \lceil \frac{r-1}{2} \rceil! \cdot \lfloor \frac{r-1}{2} \rfloor!$ for $d = 1$.
- (iii) $T(f) \geq 8 \cdot 3^{r-8}$, for $d = 2$ and $r \geq 8$.
- (iv) $T(f) \geq 2^{r-2d-1}$, for $d \geq 2$ and $r \geq 2d+2$.

It is again an easy exercise to show that the lower bound for $d = 1$ is optimal. In general, the lower bounds might not be optimal as we assumed that there is (i) only one colored Tverberg point being (ii) the intersection point of d partition blocks of exactly d points each. Up to now, we have not seen an (uncolored) example having both properties at the same time. Assuming that the colored Tverberg point is one of the vertices of Δ_N leads to a lower bound of $8 \cdot 3^{r-7}$ resp. 2^{r-d-2} for sufficiently large r .

In Section 2 we prove Theorem 2, and Theorem 3 in Section 3.

2 Proof of Theorem 2

We first prove Property (i) inductively; here the key part is the base case $k = d+2$. In a second step, we show that Property (i) implies Properties (iii) and (iv).

Let us fix some notation needed in the proof of the base case of Property (i). In our proof, we use an approach similar to the uncolored case in Hell [5]. One of our points will be moved while all the others remain fixed. During this moving process, we will keep track of the parity for the number of colored Birch partitions. In the following, we assume $d \geq 2$.

Let $k \geq 2$, fix p to be the origin $o \in \mathbb{R}^d$, and assume without restriction that all $k(d+1)$ colored points of X are on the unit sphere $S^{d-1} \subset \mathbb{R}^d$. If all points are clustered around the north pole of S^{d-1} , then $\text{cBP}_k(X) = 0$, as the origin is not in the convex hull of X . Below we do the following: We move one colored point q while fixing all others. It is sufficient to show that the parity of $\text{cBP}_k(X)$ does not change during this procedure.

Let q be a point of X . Instead of looking at q , we follow its antipode $-q$ as for any d -element subset $S \subset X \setminus \{q\}$, one has:

$$o \in \text{conv}(S \cup \{q\}) \iff -q \in \text{cone}(S).$$

From now on, we focus on all d -element subsets $S \subset X$ such that $S \cup \{q\}$ is rainbow. Every d -element subset S defines a cone in \mathbb{R}^d , all these cones decompose the sphere $S^{d-1} \subset \mathbb{R}^d$ into cells. As long as $-q$ moves inside one of these cells, $\text{cBP}_k(X)$ does not change. At some point, we are forced to move $-q$ from one cell to another. At that point $\text{cBP}_k(X)$ might change. A boundary hyperplane of a cell is defined through a $(d-1)$ -element subset $H \subset S$.

Note that our moving procedure can be chosen so that our cell decomposition is nice, and that $-q$ crosses a boundary hyperplane of the cell in a transversal way. Before looking at Birch partitions, let's look at the set \mathcal{A} of all rainbow d -simplices containing the origin. If $-q$ crosses a hyperplane defined through a subset H , then \mathcal{A} might change. Let $H' = H \cup \{q\}$. For all simplices that do not contain H' as a face, nothing changes. For the other simplices Δ the following property switches:

$$o \in \text{conv}(\Delta) \text{ before the crossing.} \iff o \notin \text{conv}(\Delta) \text{ afterwards.} \quad (1)$$

A colored Birch partition of X consists of k disjoint rainbow d -simplices containing the origin. If $-q$ crosses a hyperplane defined through $H \subset X$, then n_1 colored Birch partitions vanish, and n_2 new colored Birch partitions come up, where $n_1, n_2 \geq 0$. In fact, all Birch partitions, that include a simplex Δ , $H' \subset \Delta$, which contains the origin before the crossing, vanish. The new ones include a simplex Δ , $H' \subset \Delta$, which contains the origin after the crossing, but only if $X \setminus \Delta$ admits a colored Birch partition into $k-1$ partition blocks.

In our proof, we need a special case of Theorem 3.5 from Deza et al. [4].

Lemma 4. *For $d \geq 2$, and a given set X of $2(d+1)$ colored points in \mathbb{R}^d of $d+1$ different colors such that each color occurs exactly twice, the number of colored d -simplices containing the origin is even.*

We reprove Lemma 4 to make the reader familiar with the variational argument used below.

Proof. Let $X = \{0, 1, 2, \dots, 2d+1\}$ such that the points $2i, 2i+1$ are of color i , for all $0 \leq i \leq d$. Without restriction, we choose $q = 0$, and the boundary hyperplane of our cell spanned by $H = \{2, 4, \dots, 2(d-1)\}$. If $-q$ crosses the hyperplane through H , then exactly two colored d -simplices $\{0, 2, 4, \dots, 2(d-1), 2d\}$, and $\{0, 2, 4, \dots, 2(d-1), 2d+1\}$ are affected as observed in (1). In any case, the parity for the number of colored d -simplices containing the origin does not change. \square

Proof. Property (i) follows – as in the uncolored case – via induction from its base case $k = d+2$. Let $k \geq d+3$, and x a point of color 0. Let F_1, F_2, \dots, F_l be all rainbow d -simplices containing the origin, and using the point x . For every $i \in [l]$ the set $X \setminus F_i$ has an even number of colored Birch partitions by assumption. Adding up all these even numbers leads to $\text{cBP}_k(X)$.

Let $k = d+2$, and X be our set of $(d+1)(d+2)$ colored points. We will repeat the following step d times, and then we will finally apply Lemma 4 to complete our proof.

Step 1: Let q be a point of X , and the boundary hyperplane – that is crossed transversally – be spanned by a rainbow set H_1 . Assume without restriction that in $H_1 \cup \{q\}$ the d colors $0, 1, 2, \dots, d-1, \hat{d}$ show up, where \hat{d} means "omit d ". For every $s \in C_d$, the colored d -simplex $H_1 \cup \{q, s\}$ will change its property of containing the origin – as observed in (1) – so that some colored Birch partitions vanish, and new

ones come up. Again, new ones come up if the rest admits a colored Birch partition into $d + 1$ blocks. To prove the evenness of $\text{cBP}_{d+2}(X)$ it is sufficient to show that

$$\text{cBP}_{d+1}(X_1) = \sum_{s \in C_d^1} \text{cBP}_{d+1}(X_1 \setminus \{s\}) \text{ is even.} \quad (2)$$

Here, the set $X_1 = X \setminus H_1$ consists of $(d+1)^2 + 1$ points: The d new color classes C_0 to C_{d-1} are of size $d+1$, and color class C_d of size $d+2$. Therefore, the expression $\text{cBP}_{d+1}(X_1)$ stands for the sum over the $d+2$ possibilities to drop one of the $d+2$ points of color d from X_1 . Define the new color classes C_i^1 to be C_i minus the point of color i in $H_1 \cup \{q_1\}$, for $0 \leq i \leq d$.

In Step 1, we have reduced the partition parameter k from $d+2$ to $d+1$ by 1, and the number of points from $(d+1)(d+2)$ to $(d+1)^2 + 1$ by d . In repeating this step d times, we will end up with $k = d+2-d = 2$, and $(d+1)(d+2) - d^2 = 3d+2$ many points. Finally, the color class C_0^d will be of size 2, and C_1^d to C_d^d of size 3.

General Step $2 \leq i \leq d$: Assume that we have reduced our problem to the following statement. Show that

$$\text{cBP}_{d+3-i}(X_{i-1}) = \sum_{s_1 \in C_d^{i-1}, s_2 \in C_{d-1}^{i-1}, \dots, s_{i-1} \in C_{d-i+2}^{i-1}} \text{cBP}_{d+3-i}(X_{i-1} \setminus \{s_1, s_2, \dots, s_{i-1}\})$$

is even, where X_{i-1} has color classes $C_0^{i-1}, C_1^{i-1}, \dots, C_d^{i-1}$ such that $|C_j^{i-1}| = d+4-i$ for $j \geq d-i+2$, and $|C_j^{i-1}| = d+3-i$ otherwise.

Let q_i be a point of X_{i-1} , and the boundary hyperplane – that is crossed transversally – be spanned by a subset H_i of X_{i-1} such that $G_i = H_i \cup \{q_i\}$ is rainbow. We distinguish to cases:

1. $G_i \cap C_{d-i+2}^{i-1} = \emptyset$.
2. $G_i \cap C_{d-i+2}^{i-1} \neq \emptyset$.

In Case 1, we show that a pairing for the colored Birch partitions shows up: For every point $r \in C_{d-i+2}^{i-1}$, the property of containing the origin changes for the colored d -simplices $G_i \cup \{r\}$ while q_i crosses the hyperplane through H_i , due to (1). A d -simplex $G_i \cup \{r\}$ contributes to the number of colored Birch partitions if the rest admits a Birch partition into $d+2-i$ blocks. The latter property is independent of the current moving process. In fact, $G_i \cup \{r\}$ contributes a summand

$$\text{cBP}_{d+3-i}(X_{i-1} \setminus \{s_1, s_2, \dots, s_{i-2}, s_{i-1}\}),$$

where $s_1 \in C_d^{i-1}, s_2 \in C_{d-1}^{i-1}, \dots, s_{i-1} \in C_{d-i+2}^{i-1}$, and $r \neq s_{i-1}$, in a positive, or negative way. This contribution can be concretized to be

$$\text{cBP}_{d+2-i}(X_{i-1} \setminus (G_i \cup \{s_1, s_2, \dots, s_{i-1}, r\})).$$

But the same contribution shows up for the colored d -simplex $G_i \cup \{s_{i-1}\}$ in the summand

$$\text{cBP}_{d+3-i}(X_{i-1} \setminus \{s_1, s_2, \dots, s_{i-2}, r\}),$$

again in a positive, or negative way. In any case, the parity of $\text{cBP}_{d+3-i}(X_{i-1})$ remains unchanged.

In Case 2, let r be the unique point in $G_i \cap C_{d-i+2}^{i-1}$. Then all summands

$$\text{cBP}_{d+3-i}(X_{i-1} \setminus \{s_1, s_2, \dots, s_{i-2}, r\})$$

do not change, as any colored d -simplex not containing G_i is not affected.

We fix a point $s \in C_{d-i+2}^{i-1}$, such that $s \neq r$. Assume without restriction that in $H_i \cup \{q_i\}$ the d colors $0, 1, \dots, \widehat{d-i+1}, \dots, d$ show up such that $C_{d-i+1}^{i-1} \cap G_i = \emptyset$. For every point $t \in C_{d-i+1}^{i-1}$, the property of containing the origin changes for the colored d -simplex $G_i \cup \{t\}$, when q_i crosses the hyperplane through H_i . Every simplex $G_i \cup \{t\}$ contributes

$$\text{cBP}_{d+2-i}(X_{i-1} \setminus (G_i \cup \{s, t\}))$$

to $\text{cBP}_{d+3-i}(X_{i-1} \setminus \{s\})$ in a positive, or negative way. Note that the expression above is a sum.

Hence, it is sufficient for Case 2 to show that all these contributions sum up to an even number:

$$\text{cBP}_{d+2-i}(X_i) = \sum_{s_1 \in C_d^i, s_2 \in C_{d-1}^i, \dots, s_i \in C_{d-i+1}^i} \text{cBP}_{d+2-i}(X_i \setminus \{s_1, s_2, \dots, s_i\}),$$

where $X_i = X_{i-1} \setminus G_i$. X_i has color classes C_j^i , where C_j^i is obtained from C_j^{i-1} by deleting the point of color j in G_i for all $0 \leq j \leq d$. Note that $|C_j^i| = d+3-i$, for $j \geq d-i+1$; otherwise $|C_j^i| = d+2-i$.

Case 2 of Step i reduces our original problem in the following way: The parameter $k = d+3-i$ is reduced by 1 to $k = d-2+i$, and the number of points is reduced by d .

After d steps, the outcome of this procedure is a colored set X_d with the color class C_0^d of size 2, and color classes C_1^d to C_d^d of size 3. It remains to prove that

$$\text{cBP}_2(X_d) = \sum_{s_1 \in C_d^d, s_2 \in C_{d-1}^d, \dots, s_d \in C_1^d} \text{cBP}_2(X_d \setminus \{s_1, s_2, \dots, s_d\}) \text{ is even.}$$

For this, let q_{d+1} be a point of X_d , and the boundary hyperplane – that is crossed transversally – be spanned by a subset H_{d+1} of X_d such that $G_{d+1} = H_{d+1} \cup \{q_{d+1}\}$ is rainbow. We distinguish to cases

1. $G_{d+1} \cap C_0^d = \emptyset$.
2. $G_{d+1} \cap C_0^d \neq \emptyset$.

In Case 1, a pairing shows up as in the previous steps. Analogously, Case 2 reduces to the statement of Lemma 4. \square

Proof. Property (i) implies Properties (iii) and (iv): For $d \geq 2$, let us first prove

$$\text{cBP}_k(X) > 0 \implies \text{cBP}_k(X) \geq 2^{k-d-1} \text{ for } d \geq 2 \text{ and } k \geq d+2, \quad (3)$$

via an induction on $k \geq d+2$. This settles Property (iv).

Property (i) implies the base case $k = d+2$:

$$\text{cBP}_k(X) > 0 \implies \text{cBP}_k(X) \geq 2 = 2^{k-d-1}.$$

Let now $k \geq d+3$, and be $\text{cBP}_k(X) > 0$. Then there is a colored Birch partition F_1, F_2, \dots, F_k of X . For $1 \leq i \leq k$, let x_i be the point of color 0 such that $x_i \in F_i$. Note that for any non-empty subset I of the index set $[k]$, the set $\bigcup_{i \in I} F_i$ has again a colored Birch partition.

Using the base case for $\bigcup_{i \in [4]} F_i$, we obtain a second colored Birch partition F'_1, F'_2, F'_3, F'_4 such that $x_i \in F'_i$ for all $i \in [4]$. Without loss of generality, we can assume $F_1 \neq F'_1$. Applying the assumption to the set $X \setminus F_1$, we obtain at

least 2^{k-d-2} colored Birch partitions of X starting with F_1 . Finally, applying the assumption to the set $X \setminus F'_1$, we obtain again at least 2^{k-d-2} Birch partitions of X starting with F'_1 . The construction of the sets F_1 and F'_1 leads to the factor of 2.

To prove Property (iii), we show in the two subsequent paragraphs that a third set F''_1 can be constructed for $d = 2$, and $k \geq 7$ so that all three sets a) contain a fixed point x , and b) are pairwise distinct. Therefore, the factor 3 shows up in the lower bound for $d = 2$ and $k \geq 7$.

For $x_1 \in F_1$, the set F'_1 can be constructed as above. Now F'_1 contains a point $y \neq x_1$ that is not in F_1 , and without loss of generality we can assume $y \in F_2$. Therefore $F_2 \neq F'_2$. The set $\{4, 5, 6, 7\}$ has $\binom{4}{2} = 6$ subsets I with two elements. For every subset $I = \{i_1, i_2\}$, we apply the base case to $F_1 \cup F_3 \cup F_{i_1} \cup F_{i_2}$ so that we obtain each time a new colored Birch partition $F_1^I, F_3^I, F_{i_1}^I, F_{i_2}^I$, such that $x_1 \in F_1^I$, $x_3 \in F_3^I$, and $x_j \in F_j^I$ for both $j \in I$. If $F_1 \neq F'_1$ for one subset I , then F'_1 and F_1^I are distinct by construction. Choosing $F''_1 = F_1^I$ completes our proof.

If $F_1 = F'_1$ for all subsets I , then we proceed as follows: For every I , there is a pair of (i, j) from $I \cup \{3\}$ so that $F_i \neq F_i^I$ and $F_j \neq F_j^I$. A pair of the form $(3, j)$ is the outcome of at most three index sets, and a pair of the form (i, j) of at most two index sets, where $i, j \in \{4, 5, 6, 7\}$. As we have in total 6 pairs of indices, one index $j \in \{3, 4, 5, 6, 7\}$ shows up in at least two pairs for two subsets I_1, I_2 . Choosing the sets $F_j, F_j^{I_1}$, and $F_j^{I_2}$ completes our proof. \square

Remark 5. *It is easy to show that Theorem 2 does not hold in the more general case of continuous maps. However, it might be true if we count preimages instead of partitions.*

3 Proof of Theorem 3

Proof. The existence of at least one colored Tverberg partitions follows from Theorem 1. In the worst case, the partition consists of d partition blocks of exactly d points each, intersecting in a single point, plus $r - d$ many $d + 1$ -element sets F_1, F_2, \dots, F_{r-d} , that all contain the intersection point in their convex hulls, here we need $r \geq d$. The first $d + 1$ colors show up exactly $r - d$ times if the unique point of color $d + 1$ does not end up in one of the F_i 's. In that case, this single point is recolored with the unique color showing up $r - d - 1$ times. In both cases, the each property of $T(f)$ follows directly from the corresponding property for colored Birch partitions for $k = r - d$. \square

Remark 6. *Any lower bound l on the number of colored Tverberg points for a given map f improves our lower bounds for the number of colored Tverberg partitions by the factor of l .*

Let us conclude this note with two related problems: 1) Show that the lower bounds of Theorem 2 are tight. 2) Come up with minimal configurations for the colored Tverberg theorem.

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